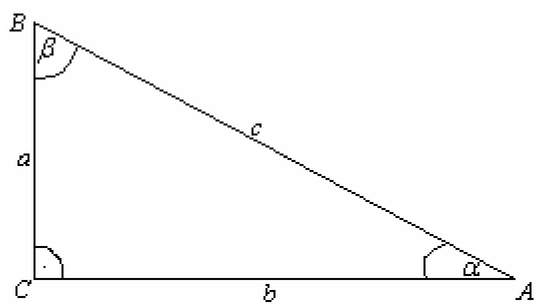
Trigonometric functions of sharp angle

Trigonometry was originally an area of mathematics that deal with calculating unknown elements of the triangle with well-known. The name comes from two Greek words , TRIGONOS-meaning triangle and METRON-meaning measures. How do we define trigonometric functions?

Observe right-angled triangle ABC.



$a, b \rightarrow$ cathetus
$c \rightarrow$ hypotenuse
$a^2 + b^2 = c^2 \rightarrow$ Pythagorean theorem

$$\sin\alpha = \frac{opposite}{hypotenuse} = \frac{a}{c}$$

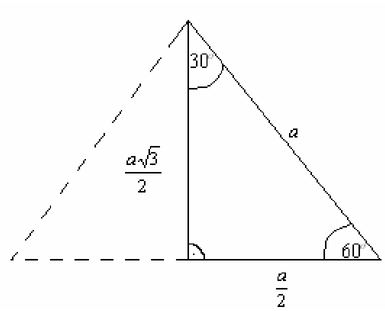
$$\cos\alpha = \frac{adjacent}{hypotenuse} = \frac{a}{c}$$

$$tg\alpha = \frac{opposite}{adjacent} = \frac{a}{b}$$

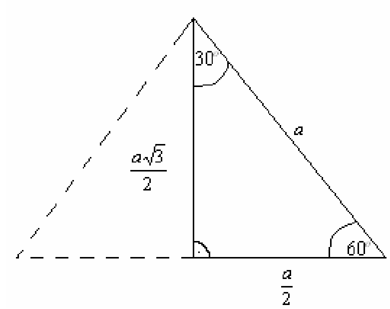
$$ctg\alpha = \frac{adjacent}{opposite} = \frac{b}{a}$$

Take heed: The symbol sin (cos, tg, ctg) will not indicate any size! Always must have angle!

How to calculate values of trigonometric functions for the angles $30^o, 45^o$ *and* 60^o ?



$$\sin 30^o = \frac{opposite}{hypotenuse} = \frac{\frac{a}{2}}{a} = \frac{a}{2a} = \frac{1}{2}$$



$$\cos 30^o = \frac{adjacent}{hypotenuse} = \frac{\frac{a\sqrt{3}}{2}}{a} = \frac{\sqrt{3}}{2}$$

$$tg 30^o = \frac{opposite}{adjacent} = \frac{\frac{a}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$ctg 30^o = \frac{adjacent}{opposite} = \frac{\frac{a\sqrt{3}}{2}}{\frac{a}{2}} = \sqrt{3}$$

Now we'll do (by definition) for angle 60^o:

$$\sin 60^o = \frac{\frac{a\sqrt{3}}{2}}{a} = \frac{\sqrt{3}}{2}$$

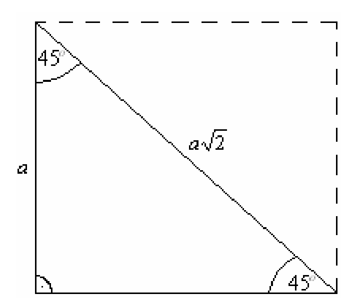
$$\cos 60^o = \frac{\frac{a}{2}}{a} = \frac{1}{2}$$

$$tg 60^o = \frac{\frac{\sqrt{3}}{2}}{\frac{a}{2}} = \frac{\sqrt{3}}{3}$$

$$ctg 60^o = \frac{\frac{a}{2}}{\frac{a\sqrt{3}}{2}} = \frac{\sqrt{3}}{3}$$

For the value of the trigonometric functions of angle 45^o , we will use half of the square.

$$\sin 45^o = \frac{opposite}{hypotenuse} = \frac{a}{a\sqrt{2}} = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$



$$\cos 45^o = \frac{adjacent}{hypotenuse} = \frac{a}{a\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$tg 45^o = \frac{opposite}{adjacent} = \frac{a}{a} = 1$$

$$ctg 45^o = \frac{adjacent}{opposite} = \frac{a}{a} = 1$$

In this way we get table:

	30^o	45^o	60^o
sinα	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
cosα	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$
tgα	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$
ctgα	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$

Of course, we will table later expand to all corners from $0^o \to 360^o$.

Basic trigonometric identity:

$$1) \sin^2\alpha + \cos^2\alpha = 1$$
$$2)\ tg\alpha = \frac{\sin\alpha}{\cos\alpha}$$
$$3)\ ctg\alpha = \frac{\cos\alpha}{\sin\alpha}$$
$$4)\ tg\alpha \cdot ctg\alpha = 1$$

To try to prove some of the identity:

1) $\sin^2\alpha + \cos^2\alpha =$ (by definitions: $\sin\alpha = \frac{a}{c}$ and $\cos\alpha = \frac{b}{c}$) $= \frac{a^2}{c^2} + \frac{b^2}{c^2} = \frac{c^2}{c^2} = 1$

2) $\frac{\sin\alpha}{\cos\alpha} = \frac{\frac{a}{c}}{\frac{b}{c}} = \frac{a \cdot c}{b \cdot c} = \frac{a}{b} = tg\alpha$

4) $tg\alpha \cdot ctg\alpha =$(change in the definition, that is $tg\alpha = \frac{a}{b}$ i $ctg\alpha = \frac{a}{b}$)$= \frac{a}{b} \cdot \frac{b}{a} = 1$

From the basic identity can be done other equality:

If you go by:

$$\sin^2\alpha + \cos^2\alpha = 1 \rightarrow \text{devide this with } \cos^2\alpha$$

$$tg\alpha = \frac{\sin^2\alpha}{\cos^2\alpha} + \frac{\cos^2\alpha}{\cos^2\alpha} = \frac{1}{\cos^2\alpha}$$

$$tg^2\alpha + 1 = \frac{1}{\cos^2\alpha} \rightarrow \text{ Express from here } \cos^2\alpha$$

$$\cos^2\alpha = \frac{1}{tg^2\alpha + 1}$$

Now this change in:

$$\sin^2\alpha + \cos^2\alpha = 1$$

$$\sin^2\alpha + \frac{1}{tg^2\alpha + 1} = 1$$

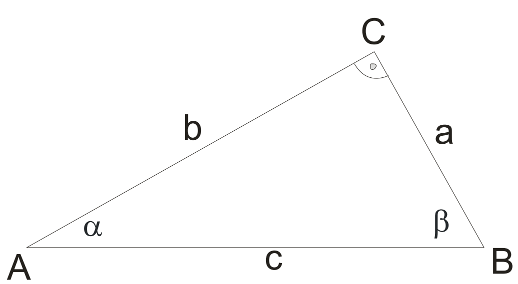
$$\sin^2\alpha = 1 - \frac{1}{tg^2\alpha + 1}$$

$$\sin^2\alpha = \frac{tg^2\alpha + 1 - 1}{tg^2\alpha + 1}$$

$$\sin^2\alpha = \frac{tg^2\alpha}{tg^2\alpha + 1}$$

These two we will remember and use them in tasks!

Next:



From the image (by definition) is:

$$\sin\alpha = \frac{a}{c} \qquad \sin\beta = \frac{b}{c}$$

$$\cos\alpha = \frac{b}{c} \qquad \cos\beta = \frac{a}{c}$$

$$tg\alpha = \frac{a}{b} \qquad tg\beta = \frac{b}{a}$$

$$ctg\alpha = \frac{b}{a} \qquad ctg\beta = \frac{a}{b}$$

We have new identity:

$$\sin(90^o - \alpha) = \cos\alpha \qquad \sin\beta = \cos\alpha$$
$$\cos(90^o - \alpha) = \sin\alpha \quad \longrightarrow \quad \cos\beta = \sin\alpha$$
$$tg(90^o - \alpha) = ctg\alpha \qquad tg\beta = ctg\alpha$$
$$ctg(90^o - \alpha) = tg\alpha \qquad ctg\beta = tg\alpha$$

1) We have cathetus of right angle triangle: a=8cm and b=6cm. Determine the value of all trigonometric functions for angles α and β.

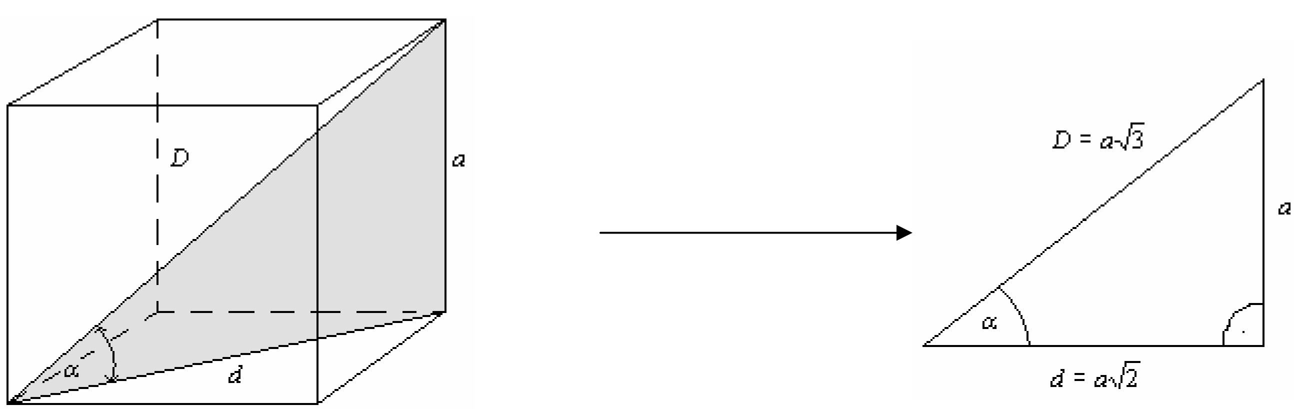
Solution:

$$a = 8cm$$
$$b = 6cm$$
$$c^2 = a^2 + b^2$$
$$c^2 = 8^2 + 6^2$$
$$c^2 = 64 + 36$$
$$c^2 = 100$$
$$c = 10cm$$

$$\sin\alpha = \frac{a}{c} = \frac{8}{10} = \frac{4}{5} = \cos\beta$$
$$\cos\alpha = \frac{b}{c} = \frac{6}{10} = \frac{3}{5} = \sin\beta$$
$$tg\alpha = \frac{a}{b} = \frac{8}{6} = \frac{4}{3} = ctg\beta$$
$$ctg\alpha = \frac{b}{a} = \frac{6}{8} = \frac{3}{4} = tg\beta$$

2) Calculate the value of trigonometric functions slope angle diagonal of the cube to basis.

Solution:



As we know , the small diagonal is $d = a\sqrt{2}$,a large (body) diagonal is: $D = a\sqrt{3}$.

By the definitions:

$$\sin\alpha = \frac{a}{a\sqrt{3}} = \frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$\cos\alpha = \frac{a\sqrt{2}}{a\sqrt{3}} = \frac{\sqrt{2}}{\sqrt{3}} = \frac{\sqrt{2}}{\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{6}}{3}$$

$$tg\alpha = \frac{a}{a\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$ctg\alpha = \frac{a\sqrt{2}}{a} = \sqrt{2}$$

3) **Calculate unknown, if:**

$c = 24cm$

$\sin\alpha = 0,8$

$a = ?$

$b = ?$

Solution:

$$\sin\alpha = \frac{a}{c}$$

$$0,8 = \frac{a}{24}$$

$$a = 24\cdot 0,8$$

$$a = 19,2cm$$

$b^2 = c^2 - a^2$ Pythagorean theorem

$$b^2 = 24^2 - (19,2)^2$$

$$b^2 = 576 - 368,64$$

$$b^2 = 207,36$$

$$b = 14,4cm$$

4) **Calculate the value of other trigonometric functions, if:**

a) $\sin\alpha = 0{,}6$

b) $\cos\alpha = \frac{12}{13}$

c) $tg\alpha = 0{,}225$

Solution:

a) $\sin\alpha = \frac{3}{5}$ because $0{,}6 = \frac{6}{10} = \frac{3}{5}$ and we will use $\sin^2\alpha + \cos^2\alpha = 1$

$$\left(\frac{3}{5}\right)^2 + \cos^2\alpha = 1$$

$$\cos^2\alpha = 1 - \frac{9}{25}$$

$$\cos^2\alpha = \frac{16}{25}$$

$$\cos\alpha = \pm\sqrt{\frac{16}{25}}$$

$$\cos\alpha = \pm\frac{4}{5}$$

As the sharp angle is in question:

$$\boxed{\cos\alpha = +\frac{4}{5}}$$

$$tg\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{\frac{3}{5}}{\frac{4}{5}} = \frac{3}{4}$$

$$ctg\alpha = \frac{1}{tg\alpha} = \frac{4}{3}$$

b)

$$\cos\alpha = \frac{12}{13}$$

$$\sin^2\alpha + \cos^2\alpha = 1$$

$$\sin^2\alpha + \left(\frac{12}{13}\right)^2 = 1$$

$$\sin^2\alpha = 1 - \frac{144}{169}$$

$$\sin^2\alpha = \frac{25}{169}$$

$$\sin\alpha = \pm\sqrt{\frac{25}{169}}$$

$$\sin\alpha = \pm\frac{5}{13}$$

$$\boxed{\sin\alpha = \frac{5}{13}}$$

$$tg\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{\frac{5}{13}}{\frac{12}{13}} = \frac{5}{12}$$

$$ctg\alpha = \frac{12}{5}$$

c) $tg\alpha = 0{,}225 = \frac{225}{1000} = \frac{9}{40}$

$$\sin^2 \alpha = \frac{tg^2\alpha}{tg^2\alpha + 1}$$

$$\sin^2 \alpha = \frac{\left(\frac{9}{40}\right)^2}{\left(\frac{9}{40}\right)^2 + 1}$$

$$\sin^2 \alpha = \frac{\frac{81}{1600}}{\frac{81}{1600} + 1}$$

$$\sin^2 \alpha = \frac{\frac{81}{1600}}{\frac{81+1600}{1600}}$$

$$\sin^2 \alpha = \frac{81}{1681}$$

$$\sin \alpha = \pm\sqrt{\frac{81}{1681}}$$

$$\sin \alpha = \pm\frac{9}{41}$$

$$\sin \alpha = +\frac{9}{41}$$

$$\cos^2 \alpha = \frac{1}{tg^2\alpha + 1}$$

$$\cos^2 \alpha = \frac{1}{\frac{1681}{1600}}$$

$$\cos^2 \alpha = \frac{1600}{1681}$$

$$\cos \alpha = \pm\sqrt{\frac{1600}{1681}}$$

$$\cos \alpha = \pm\frac{40}{41}$$

$$\cos \alpha = +\frac{40}{41}$$

$$ctg\alpha = \frac{1}{tg\alpha}$$

$$ctg\alpha = \frac{40}{9}$$

5) Calculate the value of other trigonometric functions if:

$$\text{a) } \sin\alpha = \frac{a^2-9}{a^2+9}$$

$$\text{b) } ctg\alpha = \frac{a^2-4}{4a}$$

Solution: a)

$$\sin^2\alpha + \cos^2\alpha = 1$$

$$\cos^2\alpha = 1 - \sin^2\alpha$$

$$\cos^2\alpha = 1 - \left(\frac{a^2-9}{a^2+9}\right)^2$$

$$\cos^2\alpha = 1 - \frac{(a^2-9)^2}{(a^2+9)^2}$$

$$\cos^2\alpha = \frac{(a^2+9)^2-(a^2-9)^2}{(a^2+9)^2}$$

$$\cos^2\alpha = \frac{a^4+18a^2+81-a^4+18a^2-81}{(a^2+9)^2}$$

$$\cos^2\alpha = \frac{36a^2}{(a^2+9)^2}$$

$$\cos\alpha = \sqrt{\frac{36a^2}{(a^2+9)^2}}$$

$$\cos\alpha = \frac{6a}{a^2+9}$$

$$tg\alpha = \frac{\sin\alpha}{\cos\alpha}$$

$$tg\alpha = \frac{\frac{a^2-9}{\cancel{a^2+9}}}{\frac{6a}{\cancel{a^2+9}}}$$

$$tg\alpha = \frac{a^2-9}{6a}$$

$$ctg\alpha = \frac{6a}{a^2-9}$$

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b)

$$ctg\alpha = \frac{a^2-4}{4a} \Rightarrow tg\alpha = \frac{4a}{a^2-4}$$

$$\sin^2\alpha = \frac{tg^2\alpha}{tg^2\alpha+1}$$

$$\sin^2\alpha = \frac{\left(\frac{4a}{a^2-4}\right)^2}{\left(\frac{4a}{a^2-4}\right)^2+1}$$

$$\sin^2\alpha = \frac{\frac{16a^2}{(a^2-4)^2}}{\frac{16a^2}{(a^2-4)^2}+1}$$

$$\sin^2\alpha = \frac{16a^2}{16a^2+a^4-8a^2+16}$$

$$\sin^2\alpha = \frac{16a^2}{a^4+8a^2+16}$$

$$\sin\alpha = \sqrt{\frac{16a^2}{(a^4+4)^2}}$$

$$\sin\alpha = \frac{4a}{a^2+4}$$

$$\cos^2\alpha = \frac{1}{tg^2\alpha+1}$$

$$\cos^2\alpha = \frac{1}{\left(\frac{4a}{a^2-4}\right)^2+1}$$

$$\cos^2\alpha = \frac{1}{\frac{16a^2+(a^2-4)^2}{(a^2-4)^2}}$$

$$\cos^2\alpha = \frac{1}{\frac{(a^2+4)^2}{(a^2-4)^2}}$$

$$\cos^2\alpha = \frac{(a^2-4)^2}{(a^2+4)^2}$$

$$\cos\alpha = \sqrt{\frac{(a^2-4)^2}{(a^2+4)^2}}$$

$$\cos\alpha = \frac{a^2-4}{a^2+4}$$

6) **Prove that:** $\left(1+tgx+\frac{1}{\cos x}\right)\cdot\left(1+tgx-\frac{1}{\cos x}\right)=2tgx$

Solution:

$$\left(1+tgx+\frac{1}{\cos x}\right)\cdot\left(1+tgx-\frac{1}{\cos x}\right)=$$

$$\left(1+\frac{\sin x}{\cos x}+\frac{1}{\cos x}\right)\cdot\left(1+\frac{\sin x}{\cos x}-\frac{1}{\cos x}\right)=$$

$$\frac{\cos x+\sin x+1}{\cos x}\cdot\frac{\cos x+\sin x-1}{\cos x}=$$

$$\frac{(\cos x+\sin x)^2-1^2}{\cos^2 x}=(\text{ 1 will be replaced with } \sin^2 x+\cos^2 x\text{ })$$

$$\frac{\cos^2 x+2\cos x\sin x+\sin^2 x-\sin^2 x-\cos^2 x}{\cos^2 x}=\frac{2\cancel{\cos x}\sin x}{\cos^{\cancel{2}} x}=$$

$$=2\frac{\sin x}{\cos x}=2tgx$$

7) **Prove that :** **a)** $\cos^2 18^o+\cos^2 36^o+\cos^2 54^o+\cos^2 72^o=2$

b) $tg1^o\cdot tg2^o\cdot tg3^o...tg44^o\cdot tg45^o\cdot tg46^o...tg89^o=\mathbf{1}$

Proof:

a) $\cos^2 18^o+\cos^2 36^o+\cos^2 54^o+\cos^2 72^o=2$

Because $\alpha+\beta=90^o$, $\cos\alpha=\sin\beta$, $\cos 54^o$ replace with $\sin 36^o$, and $\cos 72^o$ replace with $\sin 18^o$. Then:

$$\cos^2 18^o+\cos^2 36^o+\cos^2 54^o+\cos^2 72^o=$$

$=1+1=2$

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b)

$$tg1^o\cdot tg2^o\cdot tg3^o...tg44^o\cdot tg45^o\cdot tg46^o...tg89^o=$$

$=$ Kako je $tg\alpha=ctg\beta$ $(\alpha+\beta=90^o)$ Biće=

$$tg1^o\cdot tg2^o\cdot tg3^o...tg44^o\cdot tg45^o\cdot ctg44^o...ctg2^o\cdot ctg1^o$$

$=$ **As is**: $tg\alpha\cdot ctg\alpha=1$

$=1\cdot 1\cdot ...\cdot tg45^o=1$

8) Prove that : $\dfrac{3}{1-\sin^6\alpha-\cos^6\alpha}=(tg\alpha+ctg\alpha)^2$

Proof:

$$\frac{3}{1-\sin^6 x-\cos^6 x}=\frac{3}{1-(\sin^6 x+\cos^6 x)}=$$

We will attempt to transform expression $\sin^6 x-\cos^6 x\dots$

$$\sin^2 x-\cos^2 x=1$$

$$(A+B)^3=A^3+3A^2B+3AB^2+B^3$$

$$\sin^2 x+\cos^2 x=1/()^3$$

$$\sin^6 x+3\sin^4 x\cos^2 x+3\sin^2 x\cos^4 x+\cos^6 x=1$$

$$\sin^6 x+3\sin^2 x\cos^2 x\underbrace{(\sin^2 x+\cos^2 x)}_{1}+\cos^6 x=1$$

So: $\sin^6 x+\cos^6 x=1-3\sin^2 x\cos^2 x$

Let's go back to the task:

$$\frac{3}{1-\sin^6 x-\cos^6 x}=\frac{3}{1-(\sin^6 x+\cos^6 x)}==\frac{3}{1-1+3\sin^2 x\cos^2 x}=\frac{3}{3\sin^2 x\cos^2 x}=\frac{1}{\sin^2 x\cos^2 x}$$

To see now right side of equation:

$$\begin{aligned}(tg\alpha+ctg\alpha)^2&=tg^2\alpha+2tg\alpha ctg\alpha+ctg^2\alpha\\&=\frac{\sin^2\alpha}{\cos^2\alpha}+2+\frac{\cos^2\alpha}{\sin^2\alpha}\\&=\frac{\sin^4\alpha+2\sin^2\alpha\cos^2\alpha+\cos^4\alpha}{\sin^2\alpha\cos^2\alpha}\\&=\frac{(\sin^2\alpha+\cos^2\alpha)^2}{\sin^2\alpha\cos^2\alpha}\\&=\frac{1}{\sin^2\alpha\cos^2\alpha}\end{aligned}$$

This we have proven that the left and right side are equal.

Condition is:

$$1-\sin^6\alpha-\cos^6\alpha\neq 0$$
$$\sin^6\alpha-\cos^6\alpha\neq 1$$
$$1-3\sin^2\alpha\cos^2\alpha\neq 1$$
$$\sin^2\alpha\cos^2\alpha\neq 0$$
$$\sin\alpha\neq 0\wedge\cos\alpha\neq 0$$

9) **Prove that :** $(tg^3\alpha+\frac{1-tg\alpha}{ctg\alpha}):(\frac{1-ctg\alpha}{tg\alpha}+ctg^3\alpha)=tg^4\alpha$

Proof:

$$(tg^3\alpha+\frac{1-tg\alpha}{ctg\alpha}):(\frac{1-ctg\alpha}{tg\alpha}+ctg^3\alpha)=$$

$$(tg^3\alpha+\frac{1-tg\alpha}{\frac{1}{tg\alpha}}):(\frac{1-\frac{1}{tg\alpha}}{tg\alpha}+\frac{1}{tg^3\alpha})=$$

$$(tg^3\alpha+tg\alpha\cdot(1-tg\alpha)):(\frac{\frac{tg\alpha-1}{tg\alpha}}{tg\alpha}+\frac{1}{tg^3\alpha})=$$

$$(tg^3\alpha+tg\alpha-tg^2\alpha):(\frac{tg\alpha-1}{tg^2\alpha}+\frac{1}{tg^3\alpha})=$$

$$(tg^3\alpha-tg^2\alpha+tg\alpha):(\frac{tg\alpha(tg\alpha-1)+1}{tg^3\alpha})=$$

$$tg\alpha(tg^2\alpha-tg\alpha+1):(\frac{tg^2\alpha-tg\alpha+1}{tg^3\alpha})=\frac{tg\alpha(tg^2\alpha-tg\alpha+1)}{1}:(\frac{tg^2\alpha-tg\alpha+1}{tg^3\alpha})=$$

$$\frac{tg\alpha\,\cancel{(tg^2\alpha-tg\alpha+1)}}{1}\cdot\frac{tg^3\alpha}{\cancel{tg^2\alpha-tg\alpha+1}}=tg\alpha\cdot tg^3\alpha=\boxed{tg^4\alpha}$$

Conditions are $tg\alpha\neq 0$ **and** $ctg\alpha\neq 0$